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Finsler geodesics in the presence of a convex function and their applications

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Abstract
In this paper, we obtain a result about the existence of only a finite number of geodesics between two fixed non-conjugate points in a Finsler manifold endowed with a convex function. We apply it to Randers and Zermelo metrics. As a by-product, we also get a result about the finiteness of the number of lightlike and timelike geodesics connecting an event to a line in a standard stationary spacetime.

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1. Introduction

In this paper, we extend to Finsler metrics a result about the finiteness of the number of geodesics joining two fixed points on a Riemannian manifold, see [14]. Moreover, we present two applications of this abstract result. First we show that, under suitable assumptions, the number of lightlike or timelike geodesics with fixed arrival proper time joining an event and a timelike curve in a stationary spacetime is finite. Afterward, we study the finiteness of geodesics joining two given points in a manifold endowed with a Zermelo metric.

Let (M, F) be a non-reversible Finsler manifold; then two conditions of completeness are available: the forward and the backward completeness. As a consequence of the non-reversibility of the metric, the distance naturally associated with a Finsler metric is not symmetric. The distance $\mathrm{d}(p, q)$ between two points p and q of M is defined as the infimum of all the lengths, with respect to the Finsler structure F, of curves joining p and q on M, so it is

$$\mathrm{d}(p, q) = \inf_{\gamma\in\Omega(p,q)} \int_0^1 F(\gamma, \dot{\gamma})\,\mathrm{d}s,$$

where $\Omega(p, q)$ is the set of all the piecewise smooth curves from p to q. A *forward* (*backward*) Cauchy sequence is a sequence $\{x_i\}_{i\in\mathbb{N}}$ such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $\mathrm{d}(x_i, x_j) < \varepsilon$ for every $j > i \geqslant N$ $(i > j \geqslant N)$. The Finsler manifold (M, F) is said to be *forward (backward) complete* if all the forward (backward) Cauchy sequences converge. By the Finslerian Hopf–Rinow theorem (see [2, theorem 6.6.1 and exercise 6.6.7]) forward (backward) completeness is equivalent to forward (backward) geodesical completeness. We recall that the curve with the reverse parametrization of a geodesic for a non-reversible Finsler metric is not necessarily a geodesic. For this reason, we say that the metric is forward (backward) geodesically complete when geodesics with constant speed can be extended up to $+\infty$ (up to $-\infty$).

The main result of the paper is the following: given a forward or a backward complete Finsler manifold that admits a C^2 strictly convex function having a non-degenerate minimum point, the number of geodesics between two non-conjugate points is finite (see theorem 2.4). We will also study the existence of such convex functions for Randers, Zermelo and Fermat metrics.

Randers metrics were introduced in [23] in order to study electromagnetic fields in general relativity. Zermelo metrics were introduced in [27] to study the least time travel path of a body moving under the influence of a mild wind. Fermat metrics are a particular type of Randers metrics defined on a spacelike hypesurface of a standard stationary spacetime. They come into play in the development of a variational theory for lightlike or timelike geodesics on a standard stationary spacetime, see [6]. Such variational theory allows one to give a mathematical model for the *gravitational lensing effect* in astrophysics, see [16, 22].

Randers, Fermat and Zermelo metrics provide the same family of Finsler metrics (see, for example, [4, proposition 3.1]), but they are defined adding to a Riemannian metric on a manifold M a different geometric object, as a vector field, a positive function or a 1-form. For this reason, we study them separately. More precisely, since several results are known on the existence of convex functions in Riemannian geometry (see section 3.1), we shall study when a convex function for a Riemannian metric on a manifold M still remains convex passing to one of the Finsler structures above (see propositions 3.3, 3.11 and 3.13).

The paper is structured as follows. In section 2, we give some basic notions about Finsler geometry and obtain the main result about the existence of a finite number of geodesics joining two fixed points in the presence of a convex function (see theorem 2.4). Section 3 is devoted to applications. In subsection 3.1, we obtain a finiteness result for Randers metrics (see proposition 3.4). In subsection 3.2, we use the Fermat metric to obtain some results about the finiteness of the number of lightlike geodesics or timelike geodesics with a fixed arrival proper time, between an event and a stationary observer (see proposition 3.11). Finally, in subsection 3.3, we deduce a finiteness result for Zermelo metrics (see proposition 3.13).

2. A finiteness result in the presence of a convex function

Let M be a smooth, connected, finite-dimensional manifold and let TM be the tangent bundle of M; a non-reversible Finsler metric on M is a function $F : TM \to [0, +\infty)$ which is

(1) continuous on TM, C^∞ on $TM\setminus 0$,
(2) fiberwise positively homogeneous of degree 1, i.e. $F(x, \lambda y) = \lambda F(x, y)$, for all $x \in M$, $y \in T_xM$ and $\lambda > 0$,
(3) the square F^2 is fiberwise strictly convex, i.e. the matrix

$$g_{ij}(x, y) = \left[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)\right]$$

is positive definite for any $(x, y) \in TM\backslash 0$.

The tensor

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

is called the *fundamental tensor* of the Finsler manifold (M, F); it is a symmetric section of the tensor bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$, where $\pi^*(T^*M)$ is the dual of the pulled-back tangent bundle π^*TM over $TM\backslash 0$ (π is the projection $TM \to M$).

The *Chern connection* ∇ is the unique linear connection on π^*TM whose connection 1-forms ${\omega_j}^i$ are torsion free and almost g-compatible (see [2, theorem 2.4.1]). By using the Chern connection, one can define two different covariant derivatives $D_T W$ of a smooth vector field W along a smooth regular curve $\gamma = \gamma(s)$ on M, with velocity field $T = \dot{\gamma}$:

$$D_T W = \left(\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k {\Gamma^i}_{jk}(\gamma, T)\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} \quad \text{with reference vector } T,$$

$$D_T W = \left(\frac{\mathrm{d}W^i}{\mathrm{d}t} + W^j T^k {\Gamma^i}_{jk}(\gamma, W)\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} \quad \text{with reference vector } W,$$

where the functions ${\Gamma^i}_{jk}$ are called the components of the Chern connection ∇ and they are defined by the relation $\omega_j^i = {\Gamma^i}_{jk}\,\mathrm{d}x^k$. A geodesic of the Finsler manifold (M, F) is a smooth regular curve γ satisfying the equation

$$D_T\left(\frac{T}{F(\gamma, T)}\right) = 0,$$

with reference vector $T = \dot{\gamma}$. A curve $\gamma = \gamma(s)$ is said to have *constant speed* if $F(\gamma(s), \dot{\gamma}(s))$ is constant along γ. Geodesics are characterized as the critical points of the length and the energy functionals when considered in a suitable class of curves joining two points.

Let (M, F) be a Finsler manifold. In analogy with the Riemannian case, we say that a function $f : M \to \mathbb{R}$ is *convex* (resp. *strictly convex*) if for every constant-speed geodesic $\gamma : I \subset \mathbb{R} \to M$, $f \circ \gamma : I \to \mathbb{R}$ is convex (resp. strictly convex). Let $f : M \to \mathbb{R}$ be a C^2 function; a *critical point* $x \in M$ of f is a point where the differential of the function $\mathrm{d}f(x)$ is equal to 0. The Finslerian Hessian H_f of f is the symmetric section of the tensor bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$ over $TM\backslash 0$ given by $\nabla(\mathrm{d}f)$, where ∇ is the Chern connection associated with the Finsler metric F. In natural coordinates on $TM\backslash 0$, the Finslerian Hessian H_f of f is given by

$$(H_f)_{ij}(x, y)u^i v^j = \frac{\partial^2 f}{\partial x^i \partial x^j}(x)u^i v^j - \frac{\partial f}{\partial x^k}(x)\Gamma^k_{ij}(x, y)u^i v^j.$$

Clearly, the functions $(H_f)_{ij}$ are symmetric with respect to the indices i, j, since the components ${\Gamma^i}_{jk}$ of the connection are symmetric with respect to i, j.

If γ is a constant-speed geodesic of (M, F), then the second derivative of the function $g(s) = f(\gamma(s))$ is given by $g''(s) = (H_f)_{(\gamma(s),\dot{\gamma}(s))}(\dot{\gamma}(s), \dot{\gamma}(s))$. Thus, a C^2 function f is convex iff for every $(x, y) \in TM\backslash 0$, $(H_f)_{(x,y)}(y, y) \geqslant 0$ and it is strictly convex if $(H_f)_{(x,y)}(y, y) > 0$ (see also [26, appendix 4]).

A critical point x of f is called *non-degenerate* if $(H_f)_{(x,y)}(y, y) \neq 0$ for any $y \in T_x M$, $y \neq 0$.

The following two propositions are useful to prove the main theorem of this section.

Proposition 2.1. *Let (M, F) be a forward or a backward complete Finsler manifold and let $f : M \to \mathbb{R}$ be a C^2 convex function having a non-degenerate critical point p_0. Then p_0 is a global minimum point for f and it is the unique critical point of f.*

Proof. Let $\gamma : [a, b] \to M$ be a non-constant geodesic starting at $\gamma(a) = p_0$. Then the function $g(s) = f(\gamma(s))$ is convex in $[a, b]$, that is, $g''(s) \geqslant 0$ for any $s \in [a, b]$. As p_0 is a non-degenerate critical point, $g'(a) = 0$ and $g''(a) > 0$. Clearly, g' is an increasing function in $[a, b]$, so that $g' \geqslant 0$. Assume that there exists a point $s_0 \in]a, b]$ such that $g'(s_0) = 0$; then $g'(s) = 0$ for any $s \in [a, s_0]$, which is in contradiction with $g''(a) > 0$. Therefore, $g'(s) > 0$ for every $s \in [a, b]$; hence, $f(p_0) < f(\gamma(s))$ for any s. Now, let $q \in M$ an arbitrarily chosen point of M, by the Finslerian Hopf–Rinow theorem, there exists a geodesic $\gamma_q : [a, b] \to M$ such that $\gamma_q(a) = p_0$ and $\gamma_q(b) = q$. Since we have shown that $f(p_0) < f(\gamma_q(b)) = f(q)$ and $g'(b) > 0$, it follows that p_0 is a global minimum and the function f does not admit other critical points. □

Proposition 2.2. *Let $f : M \to \mathbb{R}$ be a C^2 convex function of a forward or a backward complete Finsler manifold (M, F) and suppose there exists a non-degenerate critical point p_0 of f (unique by proposition 2.1). Then*

$$\lim_{d(p_0,x)\to\infty} f(x) = +\infty \qquad \textit{and} \qquad \lim_{d(x,p_0)\to\infty} f(x) = +\infty.$$

Proof. By proposition 2.1, if p_0 is a non-degenerate critical point of f, then it is a global minimum and the unique critical point of f. We prove now that for any diverging sequence $(x_n)_{n\in\mathbb{N}}$ in M, it holds that $\lim_{n\to\infty} f(x_n) = +\infty$.

We assume that the Finsler manifold is forward complete (in the backward completeness case, the proof is analogous).

Let A be a normal neighborhood of p_0, i.e. there exists a star-shaped open neighborhood U of zero $U \subset T_{p_0}M$ such that $\exp_{p_0} : U \to A$ is a diffeomorphism of class C^1 in U and C^∞ in $U\backslash\{0\}$ (see [2, section 5.3]). For all $x \in A\backslash\{p_0\}$, set $u(x) = \exp_{p_0}^{-1}(x)/F\big(p_0, \exp_{p_0}^{-1}(x)\big)$, define $\gamma_x : \big]0, F\big(p_0, \exp_{p_0}^{-1}(x)\big)\big] \to M$ as $\gamma_x(s) = \exp_{p_0}(su(x))$ and

$$\phi^+(x) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=F(p_0,\exp_{p_0}^{-1}(x))} f(\gamma_x(s)).$$

Clearly, ϕ^+ is C^2 in $A\backslash\{p_0\}$ and non-negative, because convex functions have increasing derivative. Now fix $r \in \mathbb{R}$ small enough such that the sphere $S_r^+(p_0) = \{y \in M \mid \mathrm{d}(p_0, x) = r\}$ is contained in A, and define

$$\delta_0^+ = \min_{x\in S_r^+(p_0)} \phi^+(x).$$

The number δ_0^+ is positive, because otherwise there would exist $x \in S_r^+(p_0)$ such that $\phi^+(x) = 0$ and $f \circ \gamma_x$ would be a constant function, in contradiction with the hypothesis that p_0 is non-degenerate.

Now, set

$$f_0^+ = \min_{x\in S_r^+(p_0)} f(x) > -\infty.$$

Moreover, let $\gamma_n : [0, b_n] \to M$ be any minimal geodesic from p_0 to x_n having constant speed equal to 1, and let $g_n : [0, b_n] \to \mathbb{R}$ be defined as $g_n(s) = f(\gamma_n(s))$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ diverges, we can suppose that $b_n > r$, so that $p_n = \gamma_n(r)$ is well defined. By the convexity of f, we get

$$\begin{aligned} f(x_n) = g_n(b_n) &\geqslant g_n(r) + g_n'(r)(b_n - r) \\ &= f(p_n) + \phi^+(\gamma_n(r))(b_n - r) \\ &\geqslant f_0^+ + \delta_0^+(\mathrm{d}(p_0, x_n) - r) \xrightarrow{n} +\infty. \end{aligned}$$

We can analogously also prove that $\lim_{\mathrm{d}(x,p_0)\to-\infty} f(x) = +\infty$, by considering the minimizing unit-speed geodesic $\gamma_n : [c_n, 0] \to M$ from x_n to p_0, the *backward exponential* and a compact backward sphere $S_r^-(p_0)$ contained in the image of the domain of the backward exponential. The backward exponential is defined as $\exp^-_{p_0}(v) = \gamma_v(-1)$, where γ_v is the unique constant-speed geodesic with $\gamma_v(0) = p_0$ and $\dot\gamma_v(0) = v$ and

$$\phi^-(x) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=-F(p_0,(\exp^-_{p_0})^{-1}(x))} f(\gamma_x(s)),$$

where $\gamma_x : \left[-F\left(p_0, \left(\exp^-_{p_0}\right)^{-1}(x)\right), 0\right] \to M$ is now given by $\gamma_x(s) = \exp^-_{p_0}(su(x))$ and $u(x) = -\left(\exp^-_{p_0}\right)^{-1}(x)/F\left(p_0, \left(\exp^-_{p_0}\right)^{-1}(x)\right)$. Thus, we have

$$\begin{aligned} f(x_n) = g_n(c_n) &\geqslant g_n(-r) + g_n'(-r)(c_n + r)\\ &= f(p_n) + \phi^-(\gamma_n(-r))(c_n + r)\\ &= f(p_n) - \phi^-(\gamma_n(-r))\left(\mathrm{d}(x_n, p_0) - r\right)\\ &\geqslant f_0^- - \delta_0^-(\mathrm{d}(x_n, p_0) - r) \xrightarrow{n} +\infty, \end{aligned}$$

where f_0^- is the minimum value of f on $S_r^-(p_0)$ and $\delta_0^- < 0$ is the maximum value of ϕ^- on $S_r^-(p_0)$. □

Remark 2.3. Similar to [5, proposition 2.5 and lemma 2.6], it can be proved that if a manifold M admits a C^1 function f with locally Lipschitz differential, having a unique critical point which is a global minimum and having compact sublevels $f^c = \{x \in M \mid f(x) \leqslant c\}$, $c \in \mathbb{R}$, then it is contractible. So a forward or a backward complete Finsler manifold admitting a C^2 convex function having a non-degenerate minimum point is contractible (observe that in this case, the sublevels of f are compact as a consequence of the Finslerian Hopf–Rinow theorem and proposition 2.2). Apart from those in [5], other results about the topological and differentiable structures of a Riemannian manifold endowed with a (non-necessarily C^2) convex function can be found in [1, 17, 18].

Theorem 2.4. *Let (M, F) be a forward or a backward complete Finsler manifold that admits a C^2 function $f : M \to \mathbb{R}$ with positive-definite Hessian H_f everywhere and that has a minimum point. If p and q are non-conjugate points of M, then the number of geodesics in M joining p and q is finite.*

Proof. We begin by showing that a compact subset $C \subset M$ containing the image of every geodesic joining p and q does exist. Indeed, set

$$d = \max\{f(p), f(q)\},$$

since convex functions reach the maximum at the endpoints of the interval, it follows that $f(\gamma(s)) \leqslant d$ and $\gamma([0, 1]) \subseteq f^d$. By proposition 2.2 and the Finslerian Hopf–Rinow theorem, the subset $C = f^d$ is compact.

We now claim that there exists a constant E_0 such that

$$F(\gamma, \dot\gamma) \leqslant E_0, \tag{1}$$

for every geodesic $\gamma : [0, 1] \to M$ connecting p to q. To prove it by contradiction, let us assume that there exists a sequence of geodesics $\gamma_n : [0, 1] \to M$ joining p and q and having constant speed E_n with $E_n \to +\infty$ as $n \to \infty$. Consider the unit-speed geodesics $y_n : [0, E_n] \to M$ given by $y_n(s) = \gamma_n(s/E_n)$. The sequence of vectors $\{\dot y_n(0)\} \subset T_pM$ admits a subsequence converging to $v \in T_pM$. Moreover, as the images of the curves y_n are contained in C, the image of the geodesic y, such that $y(0) = p$ and $\dot y(0) = v$, is also contained in C. Since H_f is positive definite, there exists a constant $\lambda_0 = \lambda_0(C) > 0$ such that, for all $p \in C$ and all $v \in T_pM$, the following holds:

$$(H_f)_{(p,v)}(v, v) \geqslant \lambda_0 F^2(p, v).$$

So if we set $\rho(s) = f(y(s))$, then

$$\rho''(s) = (H_f)_{(y(s),\dot{y}(s))}(\dot{y}(s), \dot{y}(s)) \geqslant \lambda_0 F^2(y(s), \dot{y}(s)) = \lambda_0 > 0,$$

for every $s \in [0, +\infty)$ and hence $\lim_{s\to\infty} \rho(s) = +\infty$, which is in contradiction to the fact that the image of y is contained in the compact set C.

Now we can conclude the proof observing that if there exists an infinite number of geodesics connecting p to q, we can consider a sequence of such geodesics $\gamma_m : [0, 1] \to M$, having initial vectors $\dot{\gamma}_m(0)$. From (1), the sequence $\dot{\gamma}_m(0)$ is contained in a compact subset of T_pM; hence, it converges, up to the pass to a subsequence, to a vector $v \in T_pM$. Then, by a standard argument on the continuous dependence of solutions of ODEs with respect to initial data, the geodesics γ_m uniformly converge to the geodesic $\gamma : [0, 1] \to M$ satisfying the initial conditions $\gamma(0) = p$, $\dot{\gamma}(0) = v$. By uniform convergence, we also have $\gamma(1) = q$. Thus, the Finslerian exponential map $\exp_p$ is not injective in a neighborhood of v, in contradiction with the fact that p and q are two non-conjugate points (see [2, proposition 7.1.1]). □

Remark 2.5. It is well known that if a manifold M is noncontractible in itself (for instance, M is compact), then for any Finsler metric F on the manifold M, there exist infinitely many geodesics joining two arbitrary points p and q of M, see [6]. On the other hand, as in the Riemannian case, there are circumstances in which the number of geodesics connecting any two points on (M, F) is exactly equal to 1. For instance, the Cartan–Hadamard theorem holds for forward complete Finsler manifolds having a non-positive flag curvature; thus, if M is simply connected, the exponential map is a C^1 diffeomorphism from the tangent space at any point of M onto M (see [2, theorem 9.4.1]). Under the assumptions of theorem 2.4, the existence of infinitely many geodesics is excluded, but the existence of multiple geodesics between two points is allowed. This fact seems to be interesting in the gravitational lens effect, where a multiplicity of light rays occurs between an observer and the world line of a source, see [16, 22].

3. Applications

3.1. Randers metrics

Let (M, h) be a Riemannian manifold and let ω be a 1-form on M such that for any $x \in M$,

$$\|\omega\|_x = \sup_{v\in T_xM\setminus 0} \frac{|\omega(v)|}{\sqrt{h(v, v)}} < 1. \tag{2}$$

Then the Randers metric associated with h and ω is the Finsler metric F on M defined as

$$F(x, y) = \sqrt{h(y, y)} + \omega(y). \tag{3}$$

The couple (M, F) with F given by (3) is called *Randers manifold*. Let us observe that the condition $\|\omega\|_x < 1$, for all $x \in M$, implies not only that F is positive, but also that it has a fiberwise strongly convex square (see [2, section 11.1]).

Such type of Finsler metrics, with h Lorentzian, were considered in 1941 by Randers in a paper (see [23]) about the equivalence of relativistic electromagnetic theory (where the four-dimensional spacetime is endowed with a metric of the form (3)) and the five-dimensional Kaluza–Klein theory.

Remark 3.1. As observed in [6, remark 4.1], if the Riemannian metric (M, h) is complete and

$$\|\omega\| = \sup_{x\in M} \|\omega\|_x < 1, \tag{4}$$

then the Randers manifold (M, F) is forward and backward complete.

By using the Lévi-Cività connection ∇^h of the metric h, the geodesic equations of a Randers metric, parametrized to have constant Riemannian speed, can be written as (see [2, p 297])

$$\nabla^h_{\dot\sigma}\dot\sigma = \sqrt{h(\dot\sigma,\dot\sigma)}\hat\Omega(\dot\sigma), \tag{5}$$

where $\hat\Omega$ is the $(1, 1)$-tensor field metrically equivalent to $\Omega = \mathrm{d}\omega$, i.e. for every $(x, v) \in TM, \Omega(\cdot, v) = h(\cdot, \hat\Omega(v))$. We observe that if we define a vector field B such that $\omega(v) = h(B, v)$, then equation (5) can be expressed as

$$\nabla^h_{\dot\sigma}\dot\sigma = \sqrt{h(\dot\sigma,\dot\sigma)}\mathrm{Curl}\, B(\dot\sigma),$$

where $\mathrm{Curl}\, B(v)$ is the vector that satisfies

$$h(\mathrm{Curl}\, B(v), w) = h\big(\nabla^h_w B, v\big) - h\big(\nabla^h_v B, w\big)$$

for every v, w in T_xM.

The existence of convex functions is known for several classes of Riemannian manifolds. For instance, let $M = \mathbb{R}^N$ and let h_0 be the standard Riemannian metric on $\mathbb{R}^N$ and consider a conformally equivalent metric h to h_0, so there exists a smooth, positive function $\eta : \mathbb{R}^N \to \mathbb{R}$ such that $h = \eta(x)h_0$. Then, if

$$\eta(x) - \tfrac{3}{2}|\nabla\eta|\cdot|x| > 0,$$

where $|\cdot|$ denotes the Euclidean norm, then the function $G(x) = |x|^2$ is strictly convex for the conformal metric h (see [14, lemma 3.1]). Moreover, if (M, h) is a complete non-compact manifold having a non-negative sectional curvature, the Busemann function with changed sign is convex (see [11]). Finally, on a simply connected complete Riemannian manifold with a non-positive sectional curvature, the smooth function $x \to (\mathrm{dist}^h(x_0, x))^2$, $x_0 \in M$, is strictly convex (see [5]).

Let (M, F) be a Randers manifold, with F given by (3). Our aim is to give conditions on the associated vector field B and on the covariant differential $\nabla^h B$ ensuring that a convex function with respect to the metric Riemannian h is still convex with respect to the Randers metric F. To this end, we need to write the equation satisfied by a geodesic, parametrized with constant Randers speed, using the Lévi-Cività connection ∇^h and not with respect to the Chern connection. Since geodesics joining two fixed points are the critical points of the length functional (with respect to the Randers metric F)

$$L(\gamma) = \int_0^1 [\sqrt{h(\dot\gamma,\dot\gamma)} + \omega(\dot\gamma)]\, \mathrm{d}s,$$

they satisfy the Euler–Lagrange equations

$$\nabla^h_{\dot\sigma}(\dot\sigma/\sqrt{h(\dot\sigma,\dot\sigma)}) = \hat\Omega(\dot\sigma),$$

and, after some straightforward computations, we obtain

$$-\frac{\frac{\mathrm{d}}{\mathrm{d}s}(\sqrt{h(\dot\sigma,\dot\sigma)})}{\sqrt{h(\dot\sigma,\dot\sigma)}}\dot\sigma + \nabla^h_{\dot\sigma}\dot\sigma = \sqrt{h(\dot\sigma,\dot\sigma)}\hat\Omega(\dot\sigma).$$

If σ is parametrized with constant Randers speed $\sqrt{h(\dot\sigma,\dot\sigma)}+\omega(\dot\sigma)$, we can replace the term $\frac{\mathrm{d}}{\mathrm{d}s}(\sqrt{h(\dot\sigma,\dot\sigma)})$ in the last equation by the term $-\frac{\mathrm{d}}{\mathrm{d}s}(\omega(\dot\sigma))$, obtaining

$$\nabla^h_{\dot\sigma}\dot\sigma = \sqrt{h(\dot\sigma,\dot\sigma)}\hat\Omega(\dot\sigma) - \frac{\frac{\mathrm{d}}{\mathrm{d}s}(\omega(\dot\sigma))}{\sqrt{h(\dot\sigma,\dot\sigma)}}\dot\sigma. \tag{6}$$

In the next proposition, we compute $(H_f)(y,y)$, for each $(x,y)\in TM\backslash 0$, using the Lévi-Cività connection of h.

Proposition 3.2. *Let $f : M \to \mathbb{R}$ be a C^2 function. For each $(x,y)\in TM\backslash 0$, we have*

$$\begin{aligned} H_f(y,y) &= H^h_f(y,y) + \sqrt{h(y,y)}h(\nabla^h f, \mathrm{Curl}\, B(y)) \\ &\quad - \frac{h(\nabla^h f, y)}{F(x,y)}\big(h\big(\nabla^h_y B, y\big) + \sqrt{h(y,y)}h(B, \mathrm{Curl}\, B(y))\big), \end{aligned} \tag{7}$$

where $\nabla^h f$ and H^h_f, respectively, denote the gradient and the Hessian of f with respect to the metric h.

Proof. Let σ be a geodesic of (M,F) parametrized with constant Randers speed and such that $\sigma(0)=x$, $\dot\sigma(0)=y$. We set $\rho(s)=f(\sigma(s))$. From equation (6), recalling that for any $(x,v)\in TM$, $\hat\Omega(v)=\mathrm{Curl}\, B(v)$, we get

$$\begin{aligned} \rho''(s) &= H^h_f(\dot\sigma,\dot\sigma) + h\big(\nabla^h f, \nabla^h_{\dot\sigma}\dot\sigma\big) \\ &= H^h_f(\dot\sigma,\dot\sigma) + h(\nabla^h f, \sqrt{h(\dot\sigma,\dot\sigma)}\mathrm{Curl}\, B(\dot\sigma)) - \frac{\frac{\mathrm{d}}{\mathrm{d}s}(h(B,\dot\sigma))}{\sqrt{h(\dot\sigma,\dot\sigma)}}h(\nabla^h f,\dot\sigma). \end{aligned} \tag{8}$$

Now observe that from (6), we get

$$\frac{\mathrm{d}}{\mathrm{d}s}(h(B,\dot\sigma)) = \frac{\sqrt{h(\dot\sigma,\dot\sigma)}}{F(\sigma,\dot\sigma)}\big(h\big(\nabla^h_{\dot\sigma}B,\dot\sigma\big) + \sqrt{h(\dot\sigma,\dot\sigma)}h(B,\mathrm{Curl}\, B(\dot\sigma))\big). \tag{9}$$

Substituting (9) into (8), we obtain (7). □

Now we give a condition which ensures that an h-convex C^2 function is also convex with respect to F. We denote by $|\cdot|$ the norm with respect to the Riemannian metric h and by $\|\cdot\|$ the corresponding norms for tensor fields on M.

Proposition 3.3. *Let $f : M \to \mathbb{R}$ be a C^2 function which is convex with respect to the Riemannian metric h. Assume that f has a strictly positive Riemannian Hessian $H^h_f = \nabla^h(\mathrm{d}f)$, i.e. there exists a strictly positive function $\lambda : M \to \mathbb{R}$ such that $H^h_f(v,v) \geqslant \lambda(x)|v|^2$, for all $(x,v)\in TM$. Moreover, assume that*

$$3\|\mathrm{d}f\|\|\nabla^h B\|/(1-|B|) < \lambda(x).$$

Then f is strictly convex with respect to the Randers metric F.

Proof. We have

$$|h(\nabla^h f, \sqrt{h(y,y)}\mathrm{Curl}\, B(y))| \leqslant 2\|\nabla^h B\|\|\mathrm{d}f\||y|^2,$$

and

$$\begin{aligned} &\left|\frac{h(\nabla^h f, y)}{F(x,y)}\big(h\big(\nabla^h_y B, y\big) + \sqrt{h(y,y)}h(B,\mathrm{Curl}\, B(y))\big)\right| \\ &\quad\leqslant \frac{1}{(1-|B|)|y|}(\|\nabla^h B\||y|^2 + 2\|\nabla^h B\||B||y|^2)\|\mathrm{d}f\||y| \\ &\quad= \|\nabla^h B\|\frac{1+2|B|}{1-|B|}\|\mathrm{d}f\||y|^2. \end{aligned}$$

Thus, from (7) we obtain

$$H_f(y, y) \geqslant \left(\lambda(x) - \frac{3\|\mathrm{d}f\|\|\nabla B\|}{1-|B|}\right)|y|^2 > 0. \qquad \square$$

From the above proposition, theorem 2.4 and remark 3.1, the following proposition also holds.

Proposition 3.4. *Let M be a smooth manifold and let F be a Randers metric on M satisfying (4) and assume that the Riemannian metric h on M is complete. Assume that there exists a C^2 function $f : M \to \mathbb{R}$ having a minimum point and a Hessian H^h_f satisfying*

$$H^h_f(v, v) \geqslant \lambda(x)|v|^2,$$

for some positive function $\lambda : M \to \mathbb{R}$ and for any $(x, v) \in TM$. If

$$\|\mathrm{d}f\|\|\nabla B\|/(1 - |B|) < \lambda(x),$$

then, for any couple p and q of non-conjugate points for (M, F), there exists only a finite number of geodesics connecting p to q with respect to the Randers metric F.

Remark 3.5. The hypothesis that the points x_0 and x_1 are non-conjugate is a reasonable assumption to have only a finite number of geodesics between two points on a Riemannian or a Finsler manifold (for example, it forbids the existence of a continuum of geodesics with endpoints x_0 and x_1). Anyway, it is not a necessary condition. Indeed, on a Randers manifold, by using *stationary-to-Randers correspondence* [8] (see also the next subsection) and some bifurcation results for lightlike geodesics in a Lorentzian manifold (see [20, proposition 13]), it can be proved that if x_0 and x_1 are conjugate along the geodesic $\gamma, \gamma : [0, 1] \to M, \gamma(0) = x_0, \gamma(1) = x_1$, then there exists a continuum $(\gamma_\varepsilon)_{\varepsilon\in[0,\varepsilon_0)}$ of geodesics, $\gamma_\varepsilon : [0, a] \to M, a > 1$, and a function $s = s(\varepsilon) : [0, \varepsilon_0) \to [0, a]$ such that $\gamma_\varepsilon(0) = x_0$, for each $\varepsilon \in [0, \varepsilon_0)$, $s(\varepsilon) \to 1$, $\dot\gamma_\varepsilon(0) \to \dot\gamma(0)$, as $\varepsilon \to 0$ and $\gamma_\varepsilon(s(\varepsilon)) = \gamma(s(\varepsilon))$.

Moreover, by Sard's theorem and the fact that conjugate points are critical values of the exponential map, we know that the set of non-conjugate points to a given point x_0 is generic in M. Again using stationary-to-Randers correspondence and a recent result about genericity of the condition for being a point and a line in a standard stationary spacetime non-conjugate (see [12, 13]), we have that both the set of all the C^2 Riemannian metrics h and the set of all the C^2 1-forms ω on M, for which two fixed distinct points $x_0,\ x_1 \in M$ are non-conjugate in the Randers manifold $(M, \sqrt{h} + \omega)$, are generic in the sets of all the bilinear forms and all the 1-forms on M, with respect to a suitable topology (in particular, such a topology implies C^2-convergence on compact subsets of M). Finally, we mention that a systematic study of the Finslerian cut locus can be found in [19].

3.2. Applications to stationary spacetimes

In this subsection, we apply the results of section 2 to the study of causal geodesics connecting a point to a timelike curve on a standard stationary Lorentzian manifold.

A *standard stationary spacetime* is a Lorentzian manifold (L, l), where L splits as a product $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0, and there exist a vector field δ and a positive function β on M such that the Lorentzian metric l on L is given by

$$l((y, \tau), (y, \tau)) = g_0(y, y) + 2g_0(\delta, y)\tau - \beta(x)\tau^2, \tag{10}$$

for any $(x, t) \in M \times \mathbb{R}$ and $(y, \tau) \in T_xM \times \mathbb{R}$. We observe that a stationary spacetime, that is, a Lorentzian manifold which admits a timelike Killing field, is standard whenever the timelike Killing field is complete and the spacetime is distinguishing (see [21]).

A curve $(x(s), t(s))$ in L is a future-pointing lightlike geodesic if and only if x is a geodesic for the Randers metric, that we call the *Fermat metric*, defined as

$$F(x, y) = \sqrt{p(\delta, y)^2 + p(y, y)} + p(\delta, y), \tag{11}$$

where $p = \frac{1}{\beta} g_0$, parametrized with constant Riemannian speed $p(\dot{x}, \dot{x}) + p(\delta, \dot{x})^2$, and t coincides, up to a constant, with the Fermat length of x (see [6, theorem 4.5]).

We need to express the equation satisfied by Fermat geodesics using the Lévi-Civitià connection of the metric p. For this reason, we denote by $|\cdot|_0$ the norm with respect to the Riemannian metric g_0 and by ∇ the Lévi-Civitià connection of g_0 or the gradient with respect to g_0, $|\cdot|_1$ and $\tilde{\nabla}$, respectively, denote the norm and the Lévi-Civitià connection of p, while $\|\cdot\|_0$ and $\|\cdot\|_1$ denote the corresponding norms of the tensor fields on M. Moreover, in this subsection, we set $h(\cdot, \cdot) = p(\delta, \cdot)^2 + p(\cdot, \cdot)$.

Lemma 3.6. *A curve γ in (M, F), F defined in (11), parametrized with constant Riemannian speed $h(\dot{\gamma}, \dot{\gamma})$, is a geodesic of (M, F) if and only if it satisfies the equation*

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = F(\gamma, \dot{\gamma})\tilde{\Omega}(\dot{\gamma}) - \frac{\mathrm{d}}{\mathrm{d}s}(p(\delta, \dot{\gamma}))\delta, \tag{12}$$

where $\tilde{\Omega}(y) = \tilde{\nabla}^*\delta(y) - \tilde{\nabla}\delta(y)$, $\tilde{\nabla}\delta(y) = \tilde{\nabla}_y\delta$ and $\tilde{\nabla}^*\delta$ is the adjoint with respect to p of $\tilde{\nabla}\delta$.

Proof. Consider the length functional of the Fermat metric

$$L(x) = \int_0^1 \left[\sqrt{p(\delta(x), \dot{x})^2 + p(\dot{x}, \dot{x})} + p(\delta(x), \dot{x})\right] \mathrm{d}s. \tag{13}$$

Let $\tilde{\nabla}$ be the Lévi-Civitià connection of the Riemannian metric p; the Euler–Lagrange equations of the functional (13) can be written as

$$-\tilde{\nabla}_{\dot{\gamma}}\left(\frac{\dot{\gamma} + p(\delta, \dot{\gamma})\delta}{\sqrt{h(\dot{\gamma}, \dot{\gamma})}}\right) + \frac{p(\delta, \dot{\gamma})\tilde{\nabla}^*\delta(\dot{\gamma})}{\sqrt{h(\dot{\gamma}, \dot{\gamma})}} + \tilde{\nabla}^*\delta(\dot{\gamma}) - \tilde{\nabla}\delta(\dot{\gamma}) = 0. \tag{14}$$

Hence, if γ is parametrized to have constant Riemannian speed, we get

$$\begin{aligned}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} &= -\tilde{\nabla}_{\dot{\gamma}}(p(\delta, \dot{\gamma})\delta) + p(\delta, \dot{\gamma})\tilde{\nabla}^*\delta(\dot{\gamma}) + \sqrt{h(\dot{\gamma}, \dot{\gamma})}(\tilde{\nabla}^*\delta(\dot{\gamma}) - \tilde{\nabla}\delta(\dot{\gamma})) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}(p(\delta, \dot{\gamma}))\delta + p(\delta, \dot{\gamma})(\tilde{\nabla}^*\delta(\dot{\gamma}) - \tilde{\nabla}\delta(\dot{\gamma})) \\ &\quad + \sqrt{h(\dot{\gamma}, \dot{\gamma})}(\tilde{\nabla}^*\delta(\dot{\gamma}) - \tilde{\nabla}\delta(\dot{\gamma})) \\ &= F(\gamma, \dot{\gamma})\tilde{\Omega}(\dot{\gamma}) - \frac{\mathrm{d}}{\mathrm{d}s}(p(\delta, \dot{\gamma})\delta.\end{aligned}$$

By computing $p(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma})$, we can easily see that any solution of (12) has constant h-Riemannian speed and it satisfies equation (14). □

Lemma 3.7. *A geodesic σ of (M, F), F defined in (11), parametrized with constant Randers speed $\sqrt{h(\dot{\sigma}, \dot{\sigma})} + p(\delta, \dot{\sigma})$ satisfies the equation*

$$\tilde{\nabla}_{\dot{\sigma}}\dot{\sigma} = F(\sigma, \dot{\sigma})\tilde{\Omega}(\dot{\sigma}) - \frac{\frac{\mathrm{d}}{\mathrm{d}s}(p(\delta, \dot{\sigma}))}{\sqrt{h(\dot{\sigma}, \dot{\sigma})}}(\dot{\sigma} + F(\sigma, \dot{\sigma})\delta). \tag{15}$$

Proof. Since σ has constant Randers speed, we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{h(\dot{\sigma}, \dot{\sigma})} = -\frac{\mathrm{d}}{\mathrm{d}s}(p(\delta, \dot{\sigma})).$$

Using this equality in (14), we obtain (15). Computing $p(\tilde{\nabla}_{\dot\sigma}\dot\sigma, \dot\sigma)$, we deduce that the solutions of (15) have constant Randers speed, so that they are solutions of the Euler–Lagrange equations (14). □

We observe that a link between the geodesics of a Randers metric and those of a stationary spacetime also exists for timelike geodesics of (L, l). Indeed, each timelike geodesic of (L, l), l as in (10), can be seen as the projection on L of a lightlike geodesic in the stationary spacetime $(\tilde L, \tilde l)$, where $\tilde L = M \times \mathbb{R} \times \mathbb{R}$ and

$$\tilde l((y, v, \tau), (y, v, \tau)) = g_0(y, y) + v^2 + 2g_0(\delta, y)\tau - \beta(x)\tau^2. \tag{16}$$

More precisely, as it was observed in [6, section 4.3], a curve $z(s) = (x(s), u(s), t(s))$ in $\tilde L$ is a lightlike geodesic if and only if $(x(s), t(s))$ is a timelike geodesic of (L, l) and $\dot u(s)$ is constant and equal to E, where $-E^2 = l((\dot x(s), \dot t(s)), (\dot x(s), \dot t(s)))$. We recall that a timelike geodesic is parametrized with respect to *proper time* if $E = 1$. As a consequence, the existence of timelike geodesics with arrival proper time equal to a given $T > 0$ and joining a point (x_0, ϱ_0) to a timelike curve $\ell(\varrho) = (x_1, \varrho)$ can be deduced from the existence of geodesics connecting $(x_0, 0)$ to (x_1, T) on the manifold $N = M \times \mathbb{R}$ endowed with the Fermat metric $\tilde F$, where $\tilde F$ is given by

$$\tilde F((x, u), (y, v)) = \sqrt{\frac{1}{\beta(x)}(g_0(y, y) + v^2) + \frac{1}{\beta(x)^2} g_0(\delta, y)^2} + \frac{1}{\beta(x)} g_0(\delta, y), \tag{17}$$

for all $((x, u), (y, v)) \in TN$. Indeed, a curve $(x, t) : [0, T] \to L$ is a future-pointing timelike geodesic of (L, l), parametrized with respect to proper time, if and only if $[0, T] \ni s \to (x(s), u(s), t(s)) \in \tilde L$ is a lightlike geodesic (and therefore $u(s) = s$ up to an initial constant). At the same time, this fact is equivalent to the requirement that the curve $[0, T] \ni s \to (x(s), u(s)) \in N$ is a geodesic of $(N, \tilde F)$, parametrized with constant Riemannian speed[3] and $t = t(s)$ equal, up to an additive constant, to the length with respect to $\tilde F$ of the curve $(x(r), u(r))$, $r \in [0, s]$.

Before stating the main result of this section, we need the following definition:

Definition 3.8. *Let (L, l) be a Lorentzian manifold, $p \in L$ and $\ell : (a, b) \to L$ a timelike curve such that $p \notin \ell((a, b))$. We say that p and γ are future lightlike (resp. T-timelike) non-conjugate, if the points p and $\gamma(1)$ (resp. $\gamma(T)$) are non-conjugate along γ, for all the future-pointing lightlike geodesics $\gamma : [0, 1] \to L$ (resp. timelike geodesics $\gamma : [0, T] \to L$ parametrized with respect to proper time) such that $\gamma(0) = p$ and $\gamma(1) \in \ell((a, b))$ (resp. $\gamma(T) \in \ell((a, b))$).*

Remark 3.9. It can be proved (see theorem 3.2 of [7]) that if (L, l) is a standard stationary spacetime, then a point $(x_0, \varrho_0) \in L$ and a curve $\ell(\varrho) = (x_1, \varrho)$, with $x_0, x_1 \in M$, $x_0 \neq x_1$, are future lightlike non-conjugate if and only if x_0 and x_1 are non-conjugate in the Randers manifold (M, F).

Remark 3.10. Analogously, if the point (x_0, ϱ_0) and the curve ℓ are future T-timelike non-conjugate, then the points $(x_0, 0)$ and (x_1, T) are non-conjugate in $(N, \tilde F)$. This can be seen by using the extended stationary spacetime $(\tilde L, \tilde l)$ and the associated Randers manifold $(N, \tilde F)$. Indeed, by the fact that any Jacobi vector field along a geodesic in $(\tilde L, \tilde l)$ has a u component which is an affine function, if the points $(x(0), t(0)) = (x_0, \varrho_0)$ and $(x(T), t(T)) \in \ell(\mathbb{R})$ are non-conjugate along any timelike geodesic $s \in [0, T] \mapsto (x(s), t(s))$ in (L, l)

[3] Note that any regular curve $\gamma : [0, T] \to M$ in a Randers manifold (M, F), parametrized with constant Randers speed, can be parametrized on the same interval $[0, T]$ with constant Riemannian speed.

parametrized with respect to proper time and connecting them, then the points $(x(0), 0, t(0))$ and $(x(T), T, t(T))$ are non-conjugate along any lightlike geodesic $(x(s), s, t(s))$ in $(\tilde{L}, \tilde{l})$ connecting them. Therefore, by [7, theorem 3.2], the points $(x_0, 0)$ and (x_1, T) are non-conjugate in $(N, \tilde{F})$.

The next proposition follows the same lines as proposition 4.7 in [15], but we point out that in the latter, there is an error in the hypotheses that δ and β have to satisfy. Thus, for the sake of clearness, we redo the proof with slight changes. In addition, we obtain a new result about the finiteness of the number of timelike geodesics parametrized with respect to proper time on a given interval.

Proposition 3.11. *Let (L, l) be a standard stationary Lorentzian manifold, with l as in (10). Assume that (M, g_0) admits a C^2 convex function $f : M \to \mathbb{R}$ with a minimum point and strictly positive-definite Hessian $H_f^{g_0}$. If*

$$\sup_{x\in M} \frac{|\delta|_0}{\sqrt{\beta(x)}} \leqslant C,$$

for some $C \in \mathbb{R}^+$, and the functions $\|\nabla\delta\|_0/\sqrt{\beta(x)}$ and $|\nabla\beta|_0/\beta(x)$ are small enough, then there exists at most a finite number of future-pointing lightlike geodesics joining the point (x_0, ϱ_0) with the curve $\ell(\varrho) = (x_1, \varrho)$, with (x_0, ϱ_0) and ℓ being future lightlike non-conjugate. Moreover if the point (x_0, ϱ_0) and the curve ℓ are future T-timelike non-conjugate, the number of future-pointing timelike geodesics from (x_0, ϱ_0) to ℓ and having arrival proper time equal to T is also at most finite.

Proof. By definition of strictly positive-definite Hessian (with respect to the metric g_0), there exists a function $\lambda_0 : M \to (0, +\infty)$ such that

$$H_f^{g_0}(v, v) \geqslant \lambda_0(x)|v|_0^2,$$

for all $x \in M$ and $v \in T_xM$. Let $x : [a, b] \to M$ be a geodesic of (M, F), F as in (11), and define $\rho(s) = f(x(s))$. Then, f is strictly convex for F if $\rho''(s) > 0$ for every geodesic x. We compute ρ'' using the Hessian of f with respect to g_0:

$$\rho''(s) = H_f^{g_0}(\dot{x}, \dot{x}) + g_0(\nabla f, \nabla_{\dot{x}}\dot{x}) \geqslant H_f^{g_0}(\dot{x}, \dot{x}) - |\nabla f|_0|\nabla_{\dot{x}}\dot{x}|_0. \tag{18}$$

From equation (15), observing that

$$|\dot{x} + F(x, \dot{x})\delta|_1 = F(x, \dot{x})\sqrt{1 + |\delta|_1^2},$$

we get

$$|\tilde{\nabla}_{\dot{x}}\dot{x}|_1 \leqslant 2F(x, \dot{x})\|\tilde{\nabla}\delta\|_1|\dot{x}|_1 + \frac{\left|\frac{\mathrm{d}}{\mathrm{d}s}(p(\dot{x}, \delta))\right|}{\sqrt{|\dot{x}|_1^2 + p(\delta, \dot{x})^2}} F(x, \dot{x})\sqrt{1 + |\delta|_1^2}. \tag{19}$$

By using equation (15) again in

$$\frac{\mathrm{d}}{\mathrm{d}s}(p(\dot{x}, \delta)) = p(\tilde{\nabla}_{\dot{x}}\dot{x}, \delta) + p(\dot{x}, \tilde{\nabla}_{\dot{x}}\delta)$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} p(\dot{x}, \delta) = \frac{\sqrt{|\dot{x}|_1^2 + p(\delta, \dot{x})^2}}{F(x, \dot{x})\left(1 + |\delta|_1^2\right)}(F(x, \dot{x})p(\tilde{\Omega}(\dot{x}), \delta) + p(\dot{x}, \tilde{\nabla}_{\dot{x}}\delta)). \tag{20}$$

Finally, substituting equation (20) into (19), by $F(x, \dot{x}) \leqslant |\dot{x}|_1(1 + 2|\delta|_1)$ and the Cauchy–Schwartz inequality, we deduce that

$$|\tilde{\nabla}_{\dot{x}}\dot{x}|_1 \leqslant |\dot{x}|_1^2\|\tilde{\nabla}\delta\|_1 H(|\delta|_1), \tag{21}$$

where

$$H(r) = \frac{1}{\sqrt{1+r^2}}\big(2(1+2r)(\sqrt{1+r^2}+r)+1\big).$$

We observe that for every couple of vector fields X, Y of the manifold M, it holds that

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{\beta}{2}((X\beta^{-1})Y + (Y\beta^{-1})X - g_0(X,Y)\nabla\beta^{-1}).$$

See, for example, [10, p 181]. Hence, after some calculations, we get

$$|\nabla_{\dot{x}}\dot{x}|_0 \leqslant |\tilde{\nabla}_{\dot{x}}\dot{x}|_0 + \frac{3|\nabla\beta|_0}{2\beta(x)}|\dot{x}|_0^2, \qquad \|\tilde{\nabla}\delta\|_0 \leqslant \|\nabla\delta\|_0 + \frac{3}{2}|\delta|_0\frac{|\nabla\beta|_0}{\beta(x)}. \tag{22}$$

As $\|\tilde{\nabla}\delta\|_1 = \|\tilde{\nabla}\delta\|_0$ and $|\tilde{\nabla}_{\dot{x}}\dot{x}|_0 = \sqrt{\beta(x)}|\tilde{\nabla}_{\dot{x}}\dot{x}|_1$, by using inequalities (21) and (22), we get

$$|\nabla_{\dot{x}}\dot{x}|_0 \leqslant |\dot{x}|_0^2\left[\left(\frac{\|\nabla\delta\|_0}{\sqrt{\beta(x)}} + \frac{3}{2}\frac{|\delta|_0}{\sqrt{\beta(x)}}\frac{|\nabla\beta|_0}{\beta(x)}\right)H\left(\frac{|\delta|_0}{\sqrt{\beta(x)}}\right) + \frac{3}{2}\frac{|\nabla\beta|_0}{\beta(x)}\right].$$

Since we have assumed that $\sup_{x\in M}\frac{|\delta|_0}{\sqrt{\beta(x)}} \leqslant C$, if $\frac{\|\nabla\delta\|_0}{\sqrt{\beta(x)}}$ and $\frac{|\nabla\beta|_0}{\beta(x)}$ are small enough, equation (18) implies that f is a strictly convex function. Moreover, let us observe that the hypothesis $\sup_{x\in M}\frac{|\delta|_0}{\sqrt{\beta(x)}} \leqslant C$, for some $C \in \mathbb{R}$ and completeness of g_0, imply forward and backward completeness of the Fermat metric (see remark 3.1 and [6, remark 4.13, equation (47)]). From remark 3.9, the points x_0 and x_1 are non-conjugate in (M, F) and then by theorem 2.4 we conclude that the number of lightlike geodesic is finite. In the case of geodesics parametrized with respect to proper time and having fixed arrival proper time T, we observe that the metric $\tilde{l}$ in (16) is a stationary Lorentzian metric with $\tilde{\delta} = (\delta, 0)$ and $\tilde{\beta}(x,u) = \beta(x)$, so that we aim to apply the first part of the theorem to $(\tilde{L}, \tilde{l})$, the point $(x_0, 0, \varrho_0)$ and the line $\mathbb{R} \ni \rho \mapsto (x_1, T, \rho) \in \tilde{L}$. It is clear that the hypotheses on $\tilde{\delta}$ and $\tilde{\beta}$ are also satisfied in this case. To show the existence of a convex function, we proceed as follows: consider a real strictly convex function $g : \mathbb{R} \to \mathbb{R}$ having a minimum point. The summation $f + g : M \times \mathbb{R} \to \mathbb{R}$ defined as $(f+g)(x,y) = f(x) + g(y)$ is a strictly convex function for the metric $g_0 + \mathrm{d}u^2$, and it has a minimum point. As in the proof of proposition 4.14 in [6], we obtain the completeness of $(N, \tilde{F})$, $\tilde{F}$ defined in (17). From remark 3.10, the points $(x_0, 0)$ and (x_1, T) are non-conjugate in $(N, \tilde{F})$ and by again applying theorem 2.4, we complete the proof. □

3.3. Zermelo's problem of navigation on Riemannian manifolds

The problem that we study in this section concerns the effects of a mild wind in a Riemannian landscape (M, g). This problem is known as *Zermelo's navigation problem* (see [27]) and it was treated by Carathéodory [9] when the background is $\mathbb{R}^2$. Shen has recently generalized it to arbitrary Riemannian backgrounds in any dimension (see [25]). Following [3], we know that if the mild wind is represented by a vector field W on M such that $|W| < 1$, for each $x \in M$ ($|\cdot|$ is the norm associated with g), the trajectories that minimize (or more generally, make stationary) the travel time are the geodesics of the metric

$$F(x,y) = \frac{\sqrt{g(W,y)^2 + |y|^2\alpha(x)}}{\alpha(x)} - \frac{g(W,y)}{\alpha(x)}, \tag{23}$$

where $\alpha(x) = 1 - |W|^2$. Metrics as in (23) are of Randers type and in [3], they are used to classify Randers metrics with a constant flag curvature, while in [24], a classification of their geodesics is obtained when W is an infinitesimal homothety. Moreover, these metrics are very similar to Fermat metrics in standard stationary spacetimes. The only difference is that the

1-form in the Randers metric has the opposite sign and there is a constraint over β, that is, $\beta(x) = 1 - |\delta|^2$. Thus, a Zermelo metric is a Fermat metric with $\beta = \alpha$ and $\delta = -W$.

Remark 3.12. If $\sup_{x\in M} |W| = \mu < 1$ and the Riemannian metric g is complete, then the Zermelo metric is also forward and backward complete. This is because from equation (47) in [6], we obtain

$$\sup_{x\in M} \|\omega_x\| \leqslant \sup_{x\in M} |W(x)| = \mu < 1,$$

where $\|\omega\|$ is the norm of the 1-form ω with respect to the metric

$$h(y, y) = \frac{1}{\alpha(x)} g(y, y) + \frac{1}{\alpha^2(x)} g(y, W)^2.$$

As also $\frac{1}{\alpha} g$ is complete, applying remark 3.1, we deduce the completeness of the Zermelo metric.

From the above remark, the result in proposition 3.11 can also be proved for Zermelo metrics.

Proposition 3.13. *Let (M, g) be a complete Riemannian manifold, W be a vector field in M such that $\sup_{x\in M} |W(x)| = \mu < 1$ and $\alpha(x) = 1 - |W|^2$. Assume that (M, g) admits a C^2 convex function $f : M \to \mathbb{R}$ having a minimum point and a strictly positive-definite Hessian. If $\sup_{x\in M} \|\nabla W\|$ is small enough, then there exists a finite number of Zermelo geodesics joining two non-conjugate points of (M, F), with F being the Randers metric defined in (23).*

Proof. The completeness of the Zermelo metric follows from remark 3.12. For the existence of the convex function, it is enough to observe that

$$\frac{|\nabla(1 - |W|^2)|}{1 - |W|^2} = \frac{|\nabla|W|^2|}{1 - |W|^2} < \frac{2}{1 - \mu^2} |W| \|\nabla W\| < \frac{2}{1 - \mu^2} \|\nabla W\|$$

and also to apply proposition 3.11. □

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